Abstract

One of the laboratory design challenges is the balls-in-tubes experiment. In it, there are four tubes that each have a ball riding in them that is pushed up and down the tube by thrust generated by a fan. Here, we generate a simple model of a ball in a tube and show how feedback linearization allows for the application of linear control (e.g., PID control).

First, under a lift-coefficient hypothesis, assume that the thrust is proportional to the square of the voltage applied to the motor. That is, 

\[ T = C v_{in}^2 \]  

where \( T \) is the thrust generated by a van driven by voltage \( v_{in} \). So long as the output impedance of the amplifier generated \( v_{in} \) is sufficiently low, we can assume that electrical resistance effects are negligible.

Next, use an overly simple point-mass model for the ball, as shown in Figure 1.

\[ F = T - W = C v_{in}^2 - mg \]
\[ F = ma \]

Figure 1: Simple point-mass model of a ball. Thrust \( T \) drives ball of mass \( m \) (weight \( W \)) with upward acceleration \( a \).

In this model, the ball of mass \( m \) is driven upward by thrust \( T \) and pulled downward by gravity with weight \( W = mg \). So the net upward force on the ball is \( T - W \), which is equal to \( ma \) by Newton’s second law, where \( a \) is the magnitude of the ball’s upward acceleration. Hence, the ball’s motion is modeled by

\[ \frac{F}{ma} = \frac{T}{mg} - \frac{W}{mg}, \]  

but \( a = \dot{v} = \ddot{x} \), where \( x \) is the ball’s relative position. Using position \( x \) as an output, Equation (2) is

\[ \begin{cases} \dot{x} = v \\ \dot{v} = C m v_{in}^2 - g \end{cases} \]  

(3)

For simplicity, force \( v_{in} \geq 0 \) and use \( v_{in} = \sqrt{u} \) where \( u \geq 0 \). Hence, Equation (3) becomes

\[ \begin{cases} \dot{x} = v \\ \dot{v} = -g + \frac{C m}{u} u \end{cases} \]  

(i.e., \( \alpha(x, v) \triangleq -g \) and \( \beta(x, v) \triangleq \frac{C m}{u} \))

(4)

This system is already in normal form. Hence, without any coordinate transformation, it is immediately clear that this second-order system has relative degree 2 when position \( x \) is used as an output. So the control

\[ u = m \left( \frac{w}{C} + g \right) \]  

(i.e., \( u = \frac{w - \alpha(x, v)}{\beta(x, v)} \))

with \( w \geq -g \) renders Equation (4) into the double-integrator LTI system

\[ \begin{cases} \dot{x} = v \\ \dot{v} = w \end{cases} \]  

(5)

The parameter \( g \) is known \((9.8 \text{ m/s/s})\), the parameter \( m \) can be measured (e.g., with a scale), and the parameter \( C \) can be estimated from system data (e.g., by analyzing the acceleration of the ball when input \( u \) is constant). So the control

\[ v_{in} = \sqrt{\frac{m}{C} (w + g)} \]  

with \( w \geq -g \)

(6)

linearizes the \( w-x \) system (and needs no feedback in this simple case). Of course, the point-mass and lift-coefficient approximations may be overly naïve for this system.