# Rotary Electrodynamics of a DC Motor: Motor as Mechanical Capacitor

Lab 2: Modeling and System Identification

ECE 758: *Control System Implementation Laboratory*

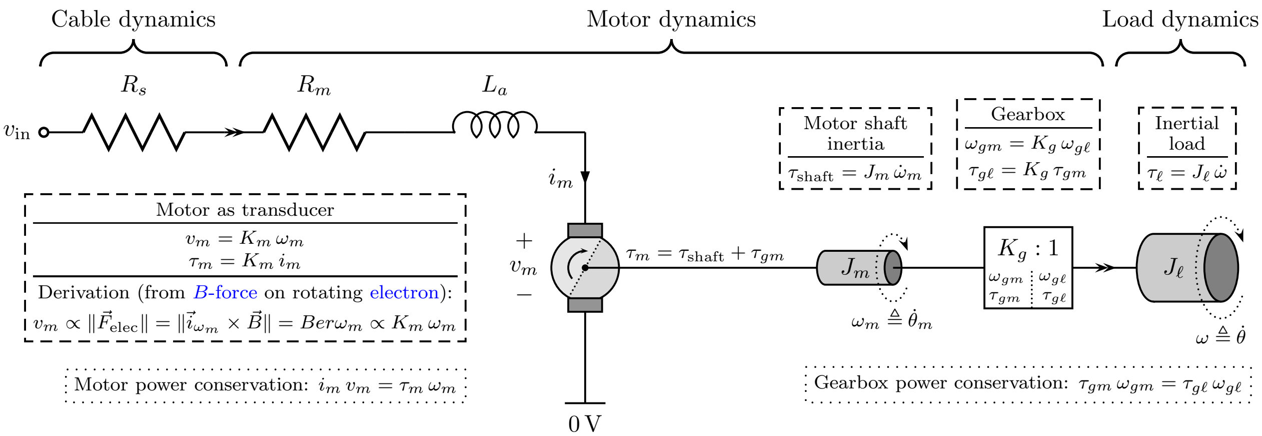


Figure 1: Basic linear model for DC motor with inertial load.

The system shown in Figure 1 is a linear model of the Quanser SRV-02 DC motor with a static inertial load. In the model,

- $R_s$ is the resistance of the cable leading to the motor. Realistically, it should also include the output impedance of the amplifier driving the cable. Because the motor is a part of the *process* being controlled, the amplifier and cable together make up the *actuator* of the *plant* (i.e., this system). Here, $R_s \approx 0\,\Omega$.
- $R_m$ is the equivalent input resistance of the motor (e.g., the wire in the coils of the motor's armature). Here, $R_m \approx 2.6\,\Omega$.
- $L_a$ is the inductance of the motor armature coils. Here, $L_a \approx 0.1\,\text{mH}$ (i.e., largely overdamped system).
- $i_m$ is the current through the motor. Because the motor is in *series* with all other electrical components, this current is shared among all components.
- $v_m$ is the voltage across the motor. As we shall see, when the motor is connected to purely inertial loads, the $i_m$–$v_m$ relationship matches that of a *capacitor*. That is, the kinetic energy stored in rotational inertia induces a proportional voltage, and the torque that is *integrated* in order to increase that kinetic energy induces a proportional current.
- $K_m$ is a proportionality constant ("motor constant") relating the motor velocity $\omega_m$ to the motor voltage $v_m$ (i.e., $v_m = K_m\omega_m$). Because power is conserved, $\tau_m = K_m i_m$ (motor efficiency is actually closer to 69%). Here, $K_m \approx 0.00767\,\text{V/rad/s}$ (but can vary greatly and is affected by viscous friction).
- $\tau_m$ is the torque being delivered by the motor to its rotational output shaft.
- $\omega_m$ is the angular speed (i.e., derivative of the motor shaft's angular position $\theta_m$) of the motor's output shaft rotations.
- $J_m$ is the rotational inertia of the motor's output shaft. Hence, $\tau_{\text{shaft}}$ is the additional torque required to rotate the output shaft at angular acceleration $\dot{\omega}_m$. Here, $J_m = 3.87 \times 10^{-7}\,\text{kg-m}^2$.
- $K_g$ is the *gear ratio* of the motor's internal *gearbox* or *transmission* system. The output of the gearbox rotates more slowly than the input, but the power transmitted across it is assumed to be perfectly conserved. Here, $K_g = 14$ (gearbox efficiency is actually closer to 85%).
- $\omega_\ell$ is the angular speed (i.e., derivative of the load's angular position $\theta$) of the rotating inertial load.
- $J_\ell$ is the rotational inertia of the load (a large disc). Hence, $\tau_\ell$ is the additional torque needed to rotate the load at angular acceleration $\dot{\omega}$. Here, $J_\ell = 3.0 \times 10^{-5}\,\text{kg-m}^2$.

This Newtonian model assumes perfectly efficient components and neglects friction and digital quantization.

# 1 Background: DC motor as linear speed transducer

Coils of wire inside a DC motor rotate through a constant magnetic field. As each coil rotates, the electrons within it are forced to cross magnetic field lines, which induces a force on each electron. The resulting force field across the coil creates an energetic difference between the electrons on opposite ends of the coil, and that energetic difference is observed as a *voltage* across the motor leads. Hence, this voltage is sometimes called an *electromotive force* (EMF) because it is a measure of the force field generated inside the motor due to its motion.

So long as the coil geometry and magnetic field are constant over the coil's rotation, the induced force (and EMF) will rise and fall linearly with the motor's angular velocity. So we assume that

$$v_m = K_m \omega_m \tag{1}$$

Because we assume that all power delivered to the motor will be transmitted to the motor output shaft,

$$i_m v_m = \tau_m \omega_m, \qquad \text{which implies} \qquad \tau_m = K_m i_m. \tag{2}$$

So we have a simple relationship between $(i_m, v_m)$ and $(\tau_m, \omega_m)$.

# 2 Equivalent shaft inertia: gearboxes as mechanical transformers

For simplicity, we will reduce Figure 1 to Figure 2.

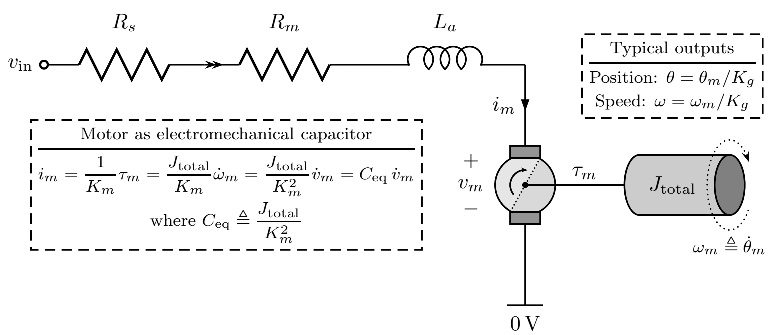


Figure 2: Basic linear model for DC motor with simplified inertial load.

In this simplified model, the motor transducer transforms the Newtonian inertial dynamics (i.e., $\tau = J\dot{\omega}$) into an equation that matches the transfer characteristics of a *capacitor*. Hence, mathematically, a motor in this configuration is identical to a capacitor with capacitance $J_{\text{total}}/K_m^2$.

**Background: rotational inertia.** Newton's second law states that an object's mass represents the proportional relationship between an applied *force* and the resulting *acceleration* (i.e., $\vec{F} = m\vec{a}$). Applying this law to a rotating object reveals that every object has a corresponding *rotational inertia* that relates an applied *torque* to a resulting *angular acceleration* (i.e., $\tau = J\alpha = J\dot{\omega}$). The moment of inertia is the rotational equivalent of mass; it encapsulates the fact that mass *at a distance* is somehow more difficult to move than nearby mass.

**Electromechanical analogs.** On a rotating shaft, all objects must rotate at the *same speed and acceleration* at the point where they connect to the shaft. However, each object requires a *different torque* for the same acceleration, and the motor's provided torque must equal the total sum of the torque used by all of the loads. Hence, there are two obvious electromechanical analogs to aid intuition in building mechanical models.

- The shaft is analogous to a *series network* of electrical components where the "current" $\omega$ is shared among all components and the "voltage" $\tau_i$ across each component sums to the total voltage $\tau$ provided to the circuit.

- The shaft is analogous to a *parallel network* of electrical components where the "voltage" $\omega$ is shared among all components and the "current" $\tau_i$ through each component sums to the total current $\tau$ provided to the circuit.

Because the motor sets up a simple relationship between speed $\omega_m$ and voltage $v_m$, it is easy to use the latter analogy. Hence, all inertial loads appear to be in parallel, and the motor can be substituted for the resulting parallel network (i.e., the analogous parallel network is in series with the other electrical components).

**Transmission as transformer.** The gearbox *steps down* the motor speed $\omega_m$ to the load speed $\omega$. It allows the motor to provide a high torque to the load without requiring a high torque (and hence, a high current) from the motor. Using the electrical analog above, the gearbox is analogous to an *electrical transformer.* In fact, just like transformers, transmissions change the torque–speed relationship so that the load on one side appears to be a scaled version of the load on the other. In particular,

$$\tau_{gm} = \frac{1}{K_g}\tau_{g\ell} = \frac{J_\ell}{K_g}\dot{\omega}_{g\ell} = \frac{J_\ell}{K_g^2}\dot{\omega}_{gm} = J_{\ell m}\dot{\omega}_{gm} \qquad \text{where} \qquad J_{\ell m} \triangleq \frac{J_\ell}{K_g^2}. \tag{3}$$

This relationship is analogous to the $Z/n^2$ impedance transformation of an electrical transformer. So

$$\tau_m = \tau_{\text{shaft}} + \tau_{gm} = J_m\dot{\omega}_m + \frac{J_\ell}{K_g^2}\dot{\omega}_m = J_{\text{total}}\,\dot{\omega}_m \qquad \text{where} \qquad J_{\text{total}} \triangleq J_m + \frac{J_\ell}{K_g^2}. \tag{4}$$

## 3 Final simplified model of a low-power DC motor

The Quanser SRV-02 DC motor is well-modeled by a first-order differential equation relating the input voltage $v_{\text{in}}$ and the output speed $\omega$. Step responses are very slow and never overshoot. In particular, the series resistance of the motor is too limiting for the motor to ever deliver great power (i.e., high energy over a short time), and this fact is reflected in its severely overdamped second-order dynamics (i.e., $R_m \gg L_a$). So we will omit the effect of $L_a$ entirely. The result the model shown in Figure 3; this model uses the quantity $J_{\text{eq}}$, which is favored by Quanser documentation.

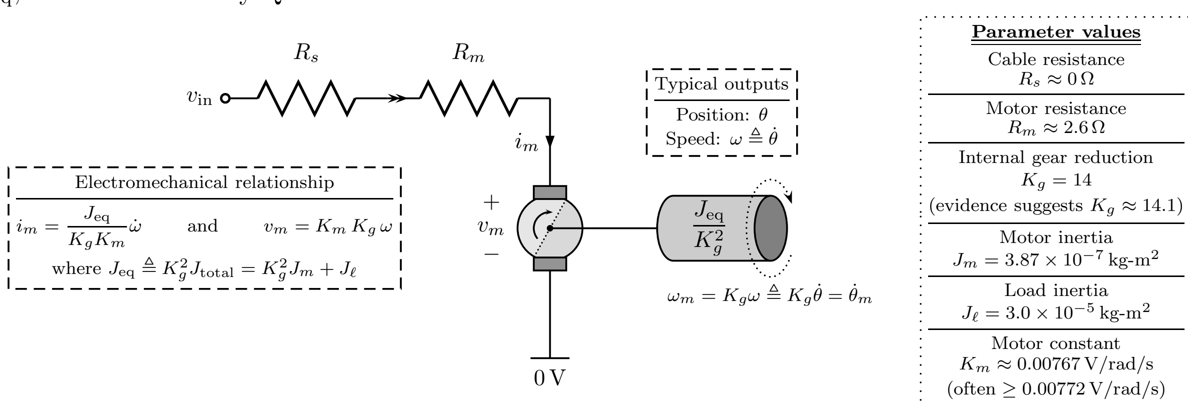


Figure 3: Simplified linear model for DC motor.

Laboratory measurements show that Quanser's $R_m$ includes both cable and motor resistance, and so $R_s \approx 0\,\Omega$ here. However, the laboratory motor is slower than predicted. This deviation may be due to inaccurate load inertia $J_\ell$ or nonlinear effects. Assuming the effect is linear, setting $R_s \approx 1.3\,\Omega$ (i.e., the actual cable resistance) restores some model accuracy even if it lacks realism.

Recall the Laplace transformations

$$\Theta(s) = \mathscr{L}\{\theta(t)\} \qquad \text{and} \qquad \Omega(s) = \mathscr{L}\{\omega(t)\}. \tag{5}$$

So the *transfer function* $\Theta(s)/V_{\text{in}}(s)$ can be found by taking the Laplace transformation of a differential equation relating position signal $\theta(t)$ and input signal $v_{\text{in}}(t)$. Likewise, the transfer function $\Omega(s)/V_{\text{in}}(s)$ can be found by taking the Laplace transformation of a differential equation relating speed signal $\omega(t)$ and input signal $v_{\text{in}}(t)$. Because speed is the derivative of position, these two transfer functions should be related in an intuitive way.

# 4 Limitations of the model: friction, quantization, and saturation

This linear model of the motor omits several characteristics that are easily observed in the laboratory.

**Friction.** The most notable omission is that of *friction* which takes several forms that can be modeled in different ways.

- *Static friction* represents a *threshold* over which the motor's output torque must cross in order to facilitate motion. This very nonlinear effect makes precise control of angular position difficult; steady-state error will be increased.
- *Kinetic friction* represents a constant torque that is always in the opposite direction of shaft rotation. It can be modeled as a *current source in parallel to the motor* with constant current whose direction matches the motor's current direction at all times. So this purely nonlinear effect can be mitigated by using high input bias or large loads (i.e., when friction "current" is negligible part of total current). The SRV-02 does have issues with kinetic friction that are observable in the laboratory.
- *Viscous friction* represents a torque that, like kinetic friction, is always in the opposite direction of shaft rotation. However, its magnitude is linearly proportional to the angular speed. This linear effect can be modeled by a *resistor in parallel to the motor.* In particular, viscous friction is the mechanical analog of *capacitor leakage.* Alternatively, its effect can be encapsulated into a Thévenin equivalent voltage source with sub-unity gain on the original source $v_{\text{in}}$, or its effect can be folded up into the $K_m$ motor constant instead.

Each of these sources of friction cause the motor to draw more current than is necessary at all speeds regardless of load. Because this extra current causes an ever present voltage drop across $R_s$ and $R_m$, the motor's top speed is reduced.

**Digital quantization.** A realistic model of this plant should also include the effects of quantization error.

- The input $v_{\text{in}}$ is generated by a *digital-to-analog converter* (DAC) with limited resolution (e.g., $2^{16}$ steps of a 20 V range from $-10$ V to 10 V). As a consequence, small movements in the input will be buried in the *quantization noise.* Each quantization step acts like a kind of threshold. When doing position control, the quantization step above and below 0 V has an identical effect as static friction, and it will have a similar effect on steady-state error.
- Measurements of the speed $\omega$ or the position $\theta$ will be conducted with an *analog-to-digital converter* (ADC) with limited resolution (e.g., a tachometer, a potentiometer, or a quadrature shaft encoder). Hence, small changes in measurements will be lost in the quantization noise. This noise will increase the steady-state error on feedback controllers.

These digital effects can be significant. In fact, the *hard edges* that they add to the signals can lead to *dangerous oscillatory components* that would otherwise not be anticipated in any purely continuous model of the system.

Of course, any digital control policy will also be discretized in time. The effects of discrete-time control on inherently continuous-time systems are vast. However, most problems are mitigated by high sampling rates (which may or may not be available depending on the available hardware and software).

**Saturation limits.** Finally, the actual $v_{\text{in}}$ source will have *saturation limits.* Here, the motor is rated for $\pm 5$ V maximum signaling, and so we must clip output signals below $-5$ V and above 5 V to prevent motor damage. As a consequence, control effort will be limited. For example, inputs of 5 V, 7 V, and 10 V will each deliver only 5 V to the input of the DC motor system, and movement in the output above that level will be lost (i.e., the system will be effectively "open loop"). This clipping can lead to problems with integral control, which will turn steady-state error over time into large output control that has no effect after it reaches a saturation limit. After saturation, steady-state error accumulates artificially fast in the integrator, and the accumulated error may lead to gross overshoot as it unwinds. Saturation limits like these are natural to legacy actuators (e.g., valves that can only open so far) as well as modern electronic actuators (e.g., outputs that reach their full-scale output values).