Name: Instructor Table number: Solution

ECE 209: *Circuits and Electronics Laboratory*

Lab 3: Operational Amplifiers and First-Order Circuits Quiz (100 points)

Description. *This quiz tests your comprehension of the introductory material on operational amplifier circuits and first-order LTI systems. This quiz is* ***closed book*** *and* ***closed notes****.*

Problem Q3-1: Operational Amplifier Circuit (20 points)

The circuit in Figure Q3-1.1 uses an ***ideal*** **operational amplifier** in a common configuration.

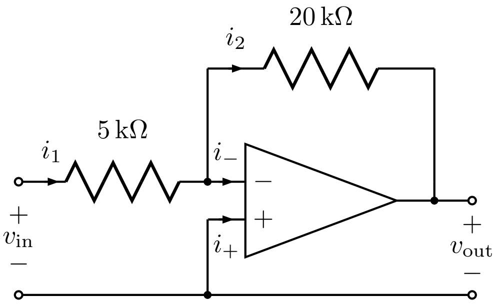


Figure Q3-1.1: Operational amplifier circuit.

The circuit produces an output v_{out} that is an amplified version of the input v_{in}. That is,

$$v_{\text{out}} = G \times v_{\text{in}}.$$

1. What is the gain G of the circuit in Figure Q3-1.1? Make sure you specify the **sign** of the gain. **(20 points)**

SOLUTION

An operational amplifier is a special kind of differential amplifier, and so its output

$$v_{\text{out}} = A \times (v_+ - v_-) .$$

That is, $(v_+ - v_-) = v_{\text{out}}/A$. However, because it is an ***ideal*** **operational** amplifier, the gain A is assumed to be practically infinite, and so if we also assume that v_{out} is finite, then $v_+ - v_- \approx 0\,\text{V}$. Further, because $v_+ = 0\,\text{V}$, then

$$v_- \approx v_+ = 0\,\text{V}.$$

Therefore,

$$i_1 \approx \frac{v_{\text{in}}}{5\,\text{k}\Omega}.$$

For an ***ideal*** **operational** amplifier, the current $i_- \approx 0\,\text{A}$, and so $i_2 \approx i_1 \approx v_{\text{in}}/(5\,\text{k}\Omega)$, and

$$\begin{aligned} v_{\text{out}} &= v_- - i_2 \times 20\,\text{k}\Omega \\ &\approx -i_1 \times 20\,\text{k}\Omega \\ &\approx -\frac{20\,\text{k}\Omega}{5\,\text{k}\Omega} v_{\text{in}} \\ &= -4 \times v_{\text{in}}. \end{aligned}$$

So gain $\boxed{G = -4.}$ The circuit in Figure Q3-1.1 is a **inverting amplifier** with a gain of magnitude 4.

Problem Q3-2: Step Response (20 points)

At time $t = 0$, a unit step is placed onto the input of a first-order linear-time-invariant (LTI) system with zero initial conditions, and the output of the system follows the trajectory shown in Figure Q3-2.1.

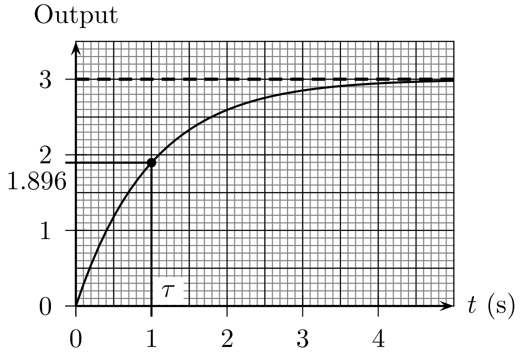


Figure Q3-2.1: Unit step response from first-order LTI system.

To answer the following questions, it may be helpful to know that

$$e^{-1} \approx 0.368 \quad \text{and} \quad 3e^{-1} \approx 1.104 \quad \text{and} \quad 1 - e^{-1} \approx 0.632 \quad \text{and} \quad 3 - 3e^{-1} \approx \boxed{1.896.}$$

2. Estimate the time constant τ of the first-order LTI system. **(10 points)**

Solution

The time constant is the time needed for the circuit to settle *within* e^{-1} (i.e., 37%) of its final value after a step input. The final value of this step response is 3, and so the time constant τ is the time when the output crosses $3 \times (1 - e^{-1})$ or 1.896. From the graph, that $\boxed{\tau \approx 1\,\text{s.}}$

3. What kind of filter is this first-order LTI system? **(10 points)**

Solution

The filter is unable to track the sharp edge of the input, but it tracks the slow constant trend of the input perfectly. Hence, it is a $\boxed{\textbf{low-pass filter.}}$

4. **BONUS:** If $H(s)$ is the transfer function of this LTI system, then what is $H(0)$? **(5 points)**

Solution

The value of $H(s)$ when $s = 0$ is the DC gain of the filter. The DC component of the input is 1, and the DC component of the output is $\sim$3, and so the DC gain is 3. That is, $\boxed{H(0) = 3.}$ Also note that by the **final value theorem** (FVT), $3 = \lim_{t \to \infty}$ "step response" $\overset{\text{FVT}}{=} \lim_{s \to 0} s(1/s)H(s) = H(0)$.

5. **BONUS:** What is the transfer function $H(s)$ of this LTI system? **(10 points)**

Solution

Every first order low-pass filter takes the form

$$K_{\text{DC}} \frac{1}{1 + \tau s} = \frac{K_{\text{DC}}}{\tau} \frac{1}{s + \frac{1}{\tau}}, \quad \text{and so because } K_{\text{DC}} = 3 \text{ and } \tau = 1, \quad \boxed{H(s) = \frac{3}{s+1}.}$$

You get the same answer if you either:

- Take the Laplace transform of the step response and multiply by s. That is, $H(s) = s \times \mathscr{L}\{3(1 - e^{-t})\} = 3s(1/s - s/(s+1)) = 3s/(s(s+1)) = 3/(s+1)$.
- Take the Laplace transform of the impulse response, which is the derivative of the step response. That is, $H(s) = \mathscr{L}\{3e^{-t}\} = 3/(s+1)$.

Problem Q3-3: Filter Analysis (60 points)

A first-order passive LTI system implemented with a resistor with resistance R and a capacitor with capacitance C is shown in Figure Q3-3.1.

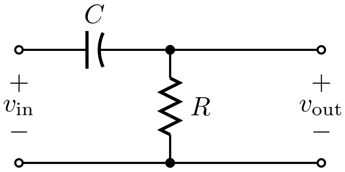


Figure Q3-3.1: First-order passive LTI system.

The following questions refer to the LTI system in Figure Q3-3.1.

6. What is the $V_{\text{out}}(s)/V_{\text{in}}(s)$ transfer function of this system? **(20 points)**

   Solution

   Recall that the capacitor's transfer characteristic is $i_C(t) = Cv_C'(t)$ where $v_C'(t)$ is the time derivative of the capacitor voltage v_C and i_C is the capacitor current. So for any $v_C(t) \triangleq \sin(\omega t)$, $i_C(t) = \omega\cos(\omega t)$. Hence, when the frequency ω is fixed, $\max\{v_C\}/\max\{i_C\} = 1/\omega$; the capacitor acts like a resistor with resistance $1/\omega$, except that it stores energy rather than dissipating it. *Mathematically* speaking, impedance acts like frequency-dependent resistance.

   So the impedance of a capacitor with capacitance C is $1/(sC)$, and the circuit is a voltage divider that divides the total impedance $R + 1/(sC)$ across the resistance R. The transfer function

   $$\boxed{H(s) \triangleq \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{R}{R+\frac{1}{sC}} = \frac{s}{s+\frac{1}{RC}} = \frac{sRC}{sRC+1}.}$$

7. What kind of filter is this system? **(20 points)**

   Solution

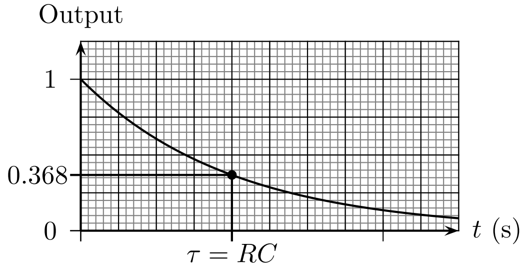
   Because $H(s)$ has a zero at $s = 0$ and a pole at $s = 1/(RC)$, then $H(0) = 0$ and $|H(j\omega)| \to 1$ as $\omega \to \infty$. So the circuit is a $\boxed{\textbf{high-pass filter.}}$

8. What is the time constant τ of this first-order LTI system? **(20 points)**

   Solution

   The time constant of a first-order LTI system is the inverse of its single pole. So $\boxed{\tau = RC.}$

9. **BONUS:** Sketch the zero-state unit step response of this first-order LTI system. Show the **time constant** τ on your sketch, and make sure the **gain** of the system is clear. **(10 points)**



Solution

Because the capacitor is initially discharged, $v_{\text{out}}(0+) = v_{\text{in}}(0+) = 1$. As it charges, the output decays as a **negative exponential** curve.

The output comes within 36.8% (which is given in Problem Q3-2) of its final value at $t = \tau = RC$.

So $v_{\text{out}}(t) = e^{-t/(RC)}$. Note that $s \times \mathscr{L}\{v_{\text{out}}\} = s/(s + (1/RC)) = H(s)$.