Sources of Phase Shift*

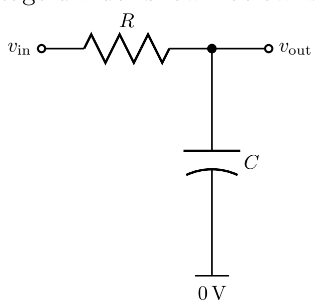
Lab 3: Operational Amplifiers and First-Order Circuits

ECE 209: *Circuits and Electronics Laboratory*

Understanding *why* phase shift occurs is one of the most difficult aspects of learning to analyze circuits. In this document, the sources of phase shift for two simple first-order filters are discussed. In both cases, the ultimate source is the *lag* between *pressure* and *flow* across the storage elements in the circuit. For example, when sinusoidal capacitor voltage is near zero, the *rate of voltage change* is very high, and so the current into the capacitor is at its peak (e.g., it is easiest to move a spring if it is relaxed). Resistive sources in the circuit turn that current into voltage, which results in two voltages in the same circuit with equal frequency and different phase. The tension between these two sources sets up different phase shifts at different frequencies.

First-Order RC Low-Pass Filter

The voltage divider shown below is a simple low-pass filter.

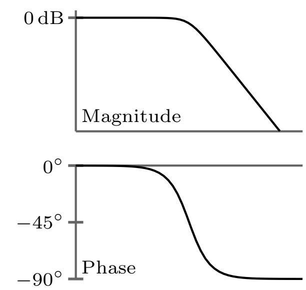


$$i_C(t) = C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} \qquad Z_C(s) = \frac{1}{sC}$$

$$H(s) \triangleq \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{sRC+1}$$

$$|H(j\omega)| = \frac{1}{\sqrt{(\omega RC)^2+1}}$$

$$\angle H(j\omega) = -\arctan(\omega RC)$$



The magnitude response of the filter can be explained by likening the capacitor to an open circuit at low frequencies (i.e., impedance magnitude $|1/(j\omega C)|$ is near infinity for $\omega \approx 0$) and a short circuit at high frequencies ($|1/(j\omega C)| \approx 0$ when $\omega \approx \infty$). That is,

- When the input is constant or has very slow changes, the capacitor acts like an open circuit. That is, it draws very little current. These slow frequencies are copied nearly perfectly onto the output.
- When the input is very quickly changing, the capacitor draws as much current as possible (i.e., it is a *short* circuit), and so there is a large voltage drop across the resistor. So these fast frequencies are nearly stripped from the output.

This open-capacitor-at-low and short-capacitor-at-high explanation of the filter's magnitude response can be extended to explain its phase response as well.

- At low frequencies, the open-circuit-like capacitor prevents any current from flowing, and so $v_{\text{out}} \approx v_{\text{in}}$, and there is no phase shift.
- At high frequencies, the short-circuit-like capacitor draws so much current across the R resistor that $v_{\text{out}} \approx 0$. In this case, we can approximate $i = v_{\text{in}}/R$. That is, **the capacitor current is in phase with the input.** However, because the capacitor voltage $v_C = \int i_C(t)/C$ and because $-\cos = \int \sin$, then the **voltage across the capacitor is a copy of the input that is *shifted* by 90°.** So the output has a $-90°$ phase shift for very high frequencies.

The phase response for intermediate frequencies smoothly connects these two extremes. In particular, the output $v_{\text{out}} = v_C = v_{\text{in}} - i_C R = v_{\text{in}} - v_C' RC$, and so $v_{\text{in}} = v_C + v_C' RC$. For any sinusoidal input, the output v_C will also be sinusoidal, and its derivative v_C' will be sinusoidal and 90° ahead in phase. Using the **trigonometric identity** that $a\sin(\phi) + b\cos(\phi) = (\sqrt{a^2+b^2})\sin(\phi + \arctan(b/a))$,

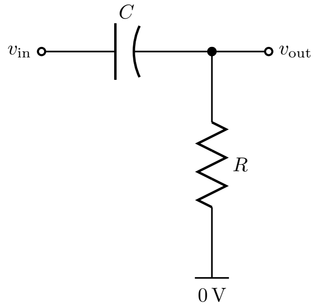
$$\left.\begin{aligned} \sin(\omega t) &= |H(j\omega)|\sin(\omega t + \angle H(j\omega)) + \omega RC|H(j\omega)|\cos(\omega t + \angle H(j\omega)) \\ &= |H(j\omega)|\left(\sqrt{(\omega RC)^2+1}\right)\sin\left(\omega t + \angle H(j\omega) + \arctan(\omega RC)\right) \end{aligned}\right\} \overset{\text{implies}}{\Longrightarrow} \begin{cases} |H(j\omega)| = \frac{1}{\sqrt{(\omega RC)^2+1}} \\ \angle H(j\omega) = -\arctan(\omega RC) \end{cases}$$

So because the input v_{in} is the *weighted* sum of two sinusoids that are 90° out of phase, the output phase shift $\angle H(j\omega) = -\arctan(\omega RC)$. As desired, $\angle H(j\omega)$ varies continuously from 0° to $-90°$ as $\omega \to \infty$.

*Document from http://www.tedpavlic.com/teaching/osu/ece209/. Source code at http://hg.tedpavlic.com/ece209/.

First-Order RC High-Pass Filter

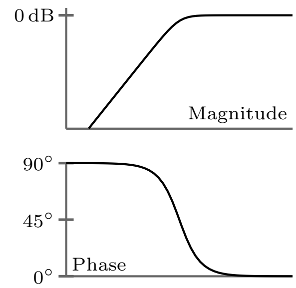
The voltage divider shown below is a simple high-pass filter.



$$i_C(t) = C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} \qquad Z_C(s) = \frac{1}{sC}$$

$$H(s) \triangleq \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{R}{R + \frac{1}{sC}} = \frac{sRC}{sRC + 1}$$

$$|H(j\omega)| = \frac{\omega RC}{\sqrt{(\omega RC)^2 + 1}}$$
$$\angle H(j\omega) = 90° - \arctan(\omega RC)$$



As before, the magnitude response of the filter can be explained by likening the capacitor to an open circuit at low frequencies (i.e., impedance magnitude $|1/(j\omega C)|$ is near infinity for $\omega \approx 0$) and a short circuit at high frequencies ($|1/(j\omega C)| \approx 0$ when $\omega \approx \infty$). That is,

- When the input is constant or has very slow changes, the capacitor acts like an open circuit. That is, it draws very little current, and so very little voltage is dropped across the output resistor. So slow frequencies on the input are nearly stripped from the output.
- When the input is very quickly changing, the capacitor draws as much current as possible (i.e., it is a *short* circuit), and so the circuit response is set primarily by the resistor. Hence, these fast frequencies are copied nearly perfectly onto the output.

This open-capacitor-at-low and short-capacitor-at-high explanation of the filter's magnitude response can be extended to explain its phase response as well.

- At low frequencies, the open-circuit-like capacitor draws so little current across the R resistor that the voltage across the capacitor is all of v_{in}. Because $i_C = Cv'_C$ and $v_C \approx v_{\text{in}}$, then the **capacitor current is in quadrature (i.e., $90°$ out of phase) with the input.** However, because the output is taken across a simple *resistor*, it is just a scaled version of the capacitor current. So **the output** has a $90°$ phase shift for low frequencies (i.e., frequencies where the capacitor is in complete control of the current).
- At high frequencies, the short-circuit-like capacitor does not restrict the current, and so $v_{\text{out}} \approx v_{\text{in}}$, and there is no phase shift.

The phase response for intermediate frequencies smoothly connects these two extremes. In particular, the output $v_{\text{out}} = v_{\text{in}} - v_C = v_{\text{in}} - (1/C)\int i_C = v_{\text{in}} - (1/(RC))\int v_{\text{out}}$, and so $v_{\text{in}} = v_{\text{out}} + (1/(RC))\int v_{\text{out}}$. For any sinusoidal input, the output will also be sinusoidal, and its integral $\int v_{\text{out}}$ will be sinusoidal and $90°$ behind in phase. Using the **trigonometric identity** that $a\sin(\phi) - b\cos(\phi) = (\sqrt{a^2 + b^2})\sin(\phi + \arctan(a/b) - 90°)$,

$$\begin{aligned}\sin(\omega t) &= |H(j\omega)|\sin(\omega t + \angle H(j\omega)) - \frac{|H(j\omega)|}{\omega RC}\cos(\omega t + \angle H(j\omega))\\ &= |H(j\omega)|\frac{\sqrt{(\omega RC)^2 + 1}}{\omega RC}\sin\left(\omega t + \angle H(j\omega) + \arctan(\omega RC) - 90°\right)\end{aligned}$$

The sinusoidal function on the right-hand side of the equation must match the sinusoidal function on the left-hand side of the equation. So

$$|H(j\omega)| = \frac{\omega RC}{\sqrt{(\omega RC)^2 + 1}} \qquad \text{and} \qquad \angle H(j\omega) = 90° - \arctan(\omega RC).$$

So because the input v_{in} is the *weighted* sum of two sinusoids that are $90°$ out of phase, the output phase shift $\angle H(j\omega) = 90° - \arctan(\omega RC)$. As desired, $\angle H(j\omega)$ varies continuously from $90°$ to $0°$ as $\omega \to \infty$.